Initial Limit Datalog: a New Extensible Class of Decidable Constrained Horn Clauses

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***Abstract*—We present *initial limit Datalog*, a new extensible class of constrained Horn clauses for which the satisfiability problem is decidable. The class may be viewed as a generalisation to higher-order logic (with a simple restriction on types) of the first-order language *limit Datalog*$_\mathbf{Z}$ (a fragment of Datalog modulo linear integer arithmetic), but can be instantiated with any suitable background theory. For example, the fragment is decidable over any countable well-quasi-order with a decidable first-order theory, such as natural number vectors under componentwise linear arithmetic, and words of a bounded, context-free language ordered by the subword relation. Formulas of initial limit Datalog have the property that, under some assumptions on the background theory, their satisfiability can be witnessed by a new kind of term model which we call *entwined structures*. Whilst the set of all models is typically uncountable, the set of all entwined structures is recursively enumerable, and model checking is decidable.**

I. Introduction

Constrained Horn Clauses (CHCs) are a class of formulas that have been found to be especially suitable for tasks in automated reasoning. They are the language of constraint logic programming [1]. More recently, there has been a concerted effort to exploit the class as a programming-language independent basis for automatic program verification [2, 3].

CHCs are a liberalisation of the class of Horn formulas in which, additionally, clauses may contain *constraints* drawn from a specified first-order *background theory*[1]. This extension preserves many of the good properties of the Horn format, such as the existence of canonical models and the sufficiency of SLD-style derivations, whilst allowing for the expression of domain-specific knowledge in the form of assertions from the background theory.

Unfortunately, this pleasing combination of expressivity and semantic characterisation comes with an algorithmic cost. In general, decidability of the satisfiability problem for a class of CHC depends on the choice of background theory, and for many theories that are typical in automated reasoning (e.g. because they are decidable), the class of CHC is undecidable. For example, [4] shows that not only is CHC over linear integer arithmetic undecidable [5], but so too CHC over complex, real or rational linear arithmetic. On the other hand, it is easy to see that CHC over the theory of equality on a finite set has decidable satisfiability.

[1]Note: in this work we will assume the background theory has a fixed interpretation, as is common in the satisfiability-modulo-theories literature.

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Since the most promising applications concern theories of infinite structures, it becomes important to identify restrictions on the format that both preserve its essential character and yet guarantee decidability. In [4], a catalogue of (sub-recursive) complexity results are derived concerning limitations placed on the use of variables within clauses and the nature of parameter passing.

An alternative approach, and the starting point for the work in this paper, is the *limit* restriction of the language *limit Datalog*$_\mathbf{Z}$, which was proposed in [6] as a foundation for declarative data analysis. Limit Datalog$_\mathbf{Z}$ can be viewed as a language of first-order CHCs over the theory of linear integer arithmetic, but with the following proviso: predicates in limit Datalog$_\mathbf{Z}$ (called *limit predicates*) are restricted so as to capture only the minimum (or maximum) numeric values in their unique integer parameter. This restriction ensures that the satisfiability problem is decidable for this class of CHC, whilst remaining expressive enough to describe important problems in data analysis (in particular, one may still describe certain kinds of recursively defined predicates over the integers).

One way to implement the limit predicate restriction is to require that all predicates with an integer parameter are either upwards or downwards closed with respect to that parameter. We enforce this by the highlighted clauses in the examples below, taken from [6].

The background theory of these examples is the combination of linear integer arithmetic and the theory of equality over a finite set. We assume that the elements of this set can be arranged into a linear order $y_1, y_2, \ldots, y_k$ (which will differ from example to example), described by two constraint formulas (i.e. of the background theory) that we will abbreviate $\mathsf{FIRST}(y_1)$ and $\mathsf{NEXT}(y_n, y_{n+1})$.

Example I.1 (Social networking)**.** In this first example, the finite set describes people who tweet and follow each other's tweets. Let us suppose we have a constraint formula[2] (i.e. of the background theory) abbreviated by $\mathsf{TH}(x, m)$, indicating the *retweet threshold*. That is, asserting that a person x will tweet a (hypothetical) message if at least m of those they follow also tweet it. Suppose we have a constraint formula $\mathsf{FOLLOWS}(x, y)$, describing when one individual x follows the tweets of another y. The following clauses constrain a proposition $\mathrm{Tw}\, x$ so that it holds if x tweeted. The proposition $\mathrm{Nt}\, x\, y\, m$ holds if, out of the people at or before y (according to the ordering on the set of people), at least m people that x

[2]One can also think more specifically of an intensional database predicate.

follows tweeted.

$$
\begin{aligned}
\mathrm{Nt}\,x\,y\,0 &\leftarrow \mathsf{FIRST}(y)\\
\mathrm{Nt}\,x\,y\,1 &\leftarrow \mathsf{FOLLOWS}(x,y) \wedge \mathsf{FIRST}(y) \wedge \mathrm{Tw}\,y\\
\mathrm{Nt}\,x\,y\,m &\leftarrow \mathrm{Nt}\,x\,y'\,m \wedge \mathsf{NEXT}(y',y)\\
\mathrm{Nt}\,x\,y\,(m+1) &\leftarrow \mathrm{Nt}\,x\,y'\,m \wedge \mathsf{FOLLOWS}(x,y)\\
&\qquad\qquad \wedge \mathsf{NEXT}(y',y) \wedge \mathrm{Tw}\,y\\
\mathrm{Nt}\,x\,y\,m &\leftarrow m \leq n \wedge \mathrm{Nt}\,x\,y\,n\\
\mathrm{Tw}\,x &\leftarrow \mathsf{TH}(x,m) \wedge \mathrm{Nt}\,x\,y\,n \wedge m \leq n
\end{aligned}
$$

Example I.2 (Path counting)**.** In this second example, the finite set describes the vertices of a directed acyclic graph and the clauses can be used to reason about the number of paths between two nodes. We assume a constraint formula $\mathsf{EDGE}(x,y)$ indicating that there is an edge from x to y.

$$
\begin{aligned}
\mathrm{Np}'\,x\,y\,z\,0 &\leftarrow \mathsf{FIRST}(z)\\
\mathrm{Np}'\,x\,y\,z\,m &\leftarrow \mathsf{FIRST}(z) \wedge \mathrm{Np}\,z\,y\,m \wedge \mathsf{EDGE}(x,z)\\
\mathrm{Np}'\,x\,y\,z\,m &\leftarrow \mathsf{NEXT}(z',z) \wedge \mathrm{Np}'\,x\,y\,z'\,m\\
\mathrm{Np}'\,x\,y\,z\,(m+n) &\leftarrow \mathsf{NEXT}(z',z) \wedge \mathrm{Np}'\,x\,y\,z'\,m\\
&\qquad\qquad \wedge \mathsf{EDGE}(x,z) \wedge \mathrm{Np}\,z\,y\,n\\
\mathrm{Np}'\,x\,y\,z\,m &\leftarrow m \leq n \wedge \mathrm{Np}'\,x\,y\,z\,n\\
\mathrm{Np}\,x\,y\,m &\leftarrow \mathrm{Np}'\,x\,y\,z\,m\\
\mathrm{Np}\,x\,x\,1 &\leftarrow \mathsf{true}\\
\mathrm{Np}\,x\,y\,m &\leftarrow m \leq n \wedge \mathrm{Np}\,x\,y\,n
\end{aligned}
$$

Here, $\mathrm{Np}'\,x\,y\,z\,m$ holds if there are at least m paths of the form $x, w, \cdots, y$ where w occurs at or before z according to the linear ordering of nodes. Finally, $\mathrm{Np}\,x\,y\,n$ holds if there are at least n paths from x to y.

Contributions

In this paper, we introduce a significant yet decidable extension of limit Datalog$_\mathbf{Z}$ which we call *initial limit Datalog*. Our language encompasses generalisations of the original work [6] along two dimensions and we define a new class of models:

(i) Parametrisation with respect to a wide range of background theories. We give a number of abstract conditions on the character of the background theory which, if satisfied, guarantee decidability of the language (Thm. III.4). Instances of particular note include all countable well-quasi orders (WQOs) with a decidable first-order theory. This contains, for example, the theory of tuples of naturals under component-wise ordering, allowing the use of predicates with more than one natural number argument.

(ii) (Un)decidability at higher type. We show that the most natural extension of limit Datalog$_\mathbf{Z}$ to higher-order logic, in which clauses can define predicates of arbitrary higher type, already has undecidable satisfiability (Thm. III.7). Through a careful analysis of the interaction between the typing discipline and model construction, we design a restriction on the types of predicates (automatically satisfied by all first-order predicates) that we call *initial*. We show that the resulting language, *initial limit Datalog*, regains decidable satisfiability (Thm. IV.3).

(iii) A recursively enumerable set of candidate models. The solution space for a given set of clauses is typically uncountable, because predicates are interpreted as subsets of the domain. A key step in proving our decidability results is to show that, remarkably, one can restrict attention to a recursively enumerable class of candidate models. To handle the higher-order case, we introduce a new representation which we call *entwined structures*, in which the interpretation of a higher *type* may depend on the interpretation of particular *terms* of lower types. They have many useful properties, and their conception is sufficiently general that we believe they may be of use for obtaining similar results beyond the scope of this paper.

Initial limit Datalog

The setting for our language is the fragment of higher-order logic known as *higher-order constrained Horn clauses* (HoCHC) [3, 7]. *Higher-order* constrained Horn clauses allow for the description of predicates of higher-types (i.e., whose subjects may themselves be predicates). Such predicates can be described by clauses built from terms of the simply typed λ-calculus when equipped with the appropriate logical constants. As in [3, 7], we forgo the use of explicit abstraction to simplify the Horn clause format.

As a first example, we demonstrate in Ex. I.3 and Ex. I.4 how the first-order limit Datalog$_\mathbf{Z}$ examples above share a common structure which can be factored out into a higher-order recursion combinator Iter of the following type:

$$\mathrm{Iter} : S \to \mathbb{Z} \to (S \to \mathbb{Z} \to o) \to o$$

Throughout (the examples of) this paper, we will use S to denote the type of a fixed finite set, o as the type of propositions and, by some abuse, $\mathbb{Z}$ as the type of the integers. This combinator can be defined as follows:

$$
\begin{aligned}
\mathrm{Iter}\,y\,n\,p &\leftarrow \mathsf{FIRST}(y) \wedge p\,y\,n\\
\mathrm{Iter}\,y\,n\,p &\leftarrow \mathsf{NEXT}(y',y) \wedge p\,y\,k \wedge \mathrm{Iter}\,y'\,m\,p \wedge n = k + m\\
\mathrm{Iter}\,y\,n\,p &\leftarrow n \leq m \wedge \mathrm{Iter}\,y\,m\,p
\end{aligned}
$$

The proposition $\mathrm{Iter}\,y\,n\,p$ describes iteration over a generic sequence of data items in S from the first item until item y, evaluating the predicate $p : S \to \mathbb{Z} \to o$ on each item and summing the associated integers to n. As in the first-order case, we must implement the limit predicate restriction, so we include the shaded clause to guarantee the (in this case) downwards closure of its integer argument.

Example I.3 (Refactoring social networking)**.** Using Iter, the whole of the social network example Ex. I.1, in which the data

items are users, can be encoded more concisely as:

$$\begin{aligned}
&\mathrm{Inc?}\, x\, y\, n \leftarrow n = 0 \vee \big(\mathsf{FOLLOWS}(x, y) \wedge \mathrm{Tw}\, y \wedge n = 1\big)\\
&\mathrm{Inc?}\, x\, y\, n \leftarrow n \leq m \wedge \mathrm{Inc?}\, x\, y\, m\\
&\quad \mathrm{Tw}\, x \leftarrow \mathsf{TH}(x, m) \wedge \mathrm{Iter}\, y\, n\, (\mathrm{Inc?}\, x) \wedge m \leq n
\end{aligned}$$

The predicate Inc?, which satisfies the limit restriction, expresses the domain specific reasoning that happens on each iteration, namely that the number of tweeters will either be increased by 0, or by 1 in case x follows some y who tweets the message.

Example I.4 (Refactoring path counting)**.** The path counting example Ex. I.2 uses a similar iterative structure. The whole example can be rewritten as:

$$\begin{aligned}
&\mathrm{NpExt}\, x\, y\, z\, m \leftarrow \big(\mathrm{Np}\, z\, y\, m \wedge \mathsf{EDGE}(x, z)\big) \vee m = 0\\
&\mathrm{NpExt}\, x\, y\, z\, m \leftarrow m \leq n \wedge \mathrm{NpExt}\, x\, y\, z\, n\\
&\quad \mathrm{Np}\, x\, y\, m \leftarrow \mathrm{Iter}\, z\, m\, (\mathrm{NpExt}\, x\, y) \vee (x = y \wedge m = 1)\\
&\quad \mathrm{Np}\, x\, y\, m \leftarrow m \leq n \wedge \mathrm{Np}\, x\, y\, n
\end{aligned}$$

In this case, the second and fourth clauses ensure that the respective predicates adhere to the limit restriction.

Example I.5 (Generic query)**.** An orthogonal benefit of higher-type predicates is to allow the expression of higher-order properties (e.g. properties of the form *for all relations* r...). Returning to Ex. I.1, the follows relation was fixed by some first-order constraint formula (or intensional database predicate) $\mathsf{FOLLOWS}(x, y)$. Using predicates of higher type, we can instead parametrise the mutually recursive predicates Nt and Tw by an *arbitrary* follows relation f of type $S \to S \to o$:

$$\begin{aligned}
\mathrm{Nt}\, x\, y\, 0\, f &\leftarrow \mathsf{FIRST}(y)\\
\mathrm{Nt}\, x\, y\, 1\, f &\leftarrow f\, x\, y \wedge \mathsf{FIRST}(y) \wedge \mathrm{Tw}\, y\, f\\
\mathrm{Nt}\, x\, y\, m\, f &\leftarrow \mathrm{Nt}\, x\, y'\, m \wedge \mathsf{NEXT}(y', y)\\
\mathrm{Nt}\, x\, y\, (m+1)\, f &\leftarrow \mathrm{Nt}\, x\, y'\, m \wedge f\, x\, y\\
&\qquad \wedge \mathsf{NEXT}(y', y) \wedge \mathrm{Tw}\, y\\
\mathrm{Nt}\, x\, y\, m\, f &\leftarrow m \leq n \wedge \mathrm{Nt}\, x\, y\, n\, f\\
\mathrm{Tw}\, x\, f &\leftarrow \mathsf{TH}(x, m) \wedge \mathrm{Nt}\, x\, y\, n\, f \wedge m \leq n
\end{aligned}$$

This allows us to check that a property of the system holds *independently* of who follows whom. For example, according to Kaminski et al.'s formulation, nobody will tweet the message if we fix all retweet thresholds at 1. To verify this, we set $\mathsf{TH}(x, m)$ to the constraint formula $m = 1$ and decide satisfiability of the clauses extended with the following goal:

$$\mathit{false} \leftarrow \mathrm{Tw}\, x\, f$$

From the satisfiability of the clauses, we can deduce that there does not exist a choice of an individual x and a followers relation f for which the message would be tweeted.

Examples I.3 to I.5 are not limit Datalog$_\mathbf{Z}$ problems, but they are problems of our generalisation: *initial limit Datalog*. As well as admitting the definition of higher-order relations, in place of the theory of integer linear arithmetic we allow for the theory of any preordered set W satisfying certain conditions.

Initial limit Datalog problem and satisfiability: Henceforth let W be a preordered set with a decidable first-order theory, such that every upwards closed subset of W is definable in the theory. We consider relational types generated from W and any finite set S (abusing notation by naming the types after their interpretations).

An ***initial limit Datalog problem*** is a (finite) set Γ of HoCHC clauses over W and S such that for every predicate $X : \rho$ in the signature, with $\rho = \sigma_1 \to \cdots \to \sigma_k \to o$ of order n (say):

(i) ρ is ***initial***, meaning $\sigma_j = W$ for at most one j, and if there is such a j then for all $i < j$, $\mathrm{order}(\sigma_i) < \mathrm{order}(\sigma_j \to \cdots \to \sigma_k \to o)$; moreover each σ_i is S, or W, or initial.

(ii) if $\sigma_j = W$ for some j, then Γ contains the ***limit clause***

$$X\, \overline{z}\, x\, \overline{z'} \;\leftarrow\; x \leq y \wedge X\, \overline{z}\, y\, \overline{z'}$$

(and ρ is called an ***active type***).

The ***satisfiability problem*** for *initial limit Datalog* asks: given an initial limit Datalog problem Γ, is it satisfiable (modulo the theory of W and S)?

We show in Sec. III-B that a naïve extension to higher order leads to undecidability, but the forgoing examples and those we will present in the sequel all obey a certain discipline in the way that the background type W and higher types interact. This is captured by the *initial* restriction, (i), which requires that the types of terms that may be captured by a partial application are of strictly lower order than the partial application itself. It is easy to verify that this condition holds for the type of Iter and one can also see it in the types of our higher-order generalisation of Nt and (trivially) Tw:

$$\begin{aligned}
\mathrm{Tw} &: S \to (S \to S \to o) \to o\\
\mathrm{Nt} &: S \to S \to \mathbb{Z} \to (S \to S \to o) \to o
\end{aligned}$$

Note that all formulas of limit Datalog$_\mathbf{Z}$ already satisfy requirements (i) and (ii); and $\mathbb{Z}$, under the theory of linear integer arithmetic, is an appropriate instantiation of W.

Parametrisation of initial limit Datalog by the type W allows for a variety of interesting background structures beyond integer linear arithmetic. For example, any countable well-quasi-ordering with a decidable background theory (which must include constants for each element of the structure) satisfies the requirements on W, such as:

a. Tuples of natural numbers, under componentwise ordering with the theory of linear arithmetic on components.

b. Words of a bounded, context-free language, under the subword order [8].

c. Basic process algebra under the subword order. BPA is an automatic structure, and so, has a decidable first-order

theory. There are other examples in the same vein, e.g., communicating finite-state machines [9].

The following example is a higher-order instance of initial limit datalog where the preorder W is the WQO of tuples of natural numbers, with the theory of linear arithmetic on components. Notice that in this case, there may be multiple consecutive parameters of type $\mathbb{N}$ in a predicate.

Example I.6 (Integration)**.** Monotone decreasing functions $\mathbb{N} \to \mathbb{N}$ can be represented by downwards closed subsets of $\mathbb{N} \times \mathbb{N}$: such a function f is uniquely identified by $\{(x, y) : y < f(x)\}$. Higher-order initial limit Datalog allows us to define a predicate which computes integrals[3] over such functions.

$$\begin{aligned}
&\mathrm{Integral} : \mathbb{N} \to \mathbb{N} \to (\mathbb{N} \to \mathbb{N} \to o) \to o \\
&\mathrm{Integral}\ tot\ bd\ f \leftarrow tot = 0 \\
&\mathrm{Integral}\ tot\ bd\ f \leftarrow tot = x + y + 1 \wedge \mathrm{Integral}\, x\, (bd + 1)\, f \\
&\qquad\qquad \wedge\ f\ bd\ y \\
&\mathrm{Integral}\ tot\ bd\ f \leftarrow tot \leq s \wedge bd \leq c \wedge \mathrm{Integral}\, s\, c\, f
\end{aligned}$$

$$\begin{aligned}
&\mathrm{Exp} : \mathbb{N} \to \mathbb{N} \to o \\
&\mathrm{Exp}\, m\, n \leftarrow m = 0 \wedge n < 128 \\
&\mathrm{Exp}\, m\, n \leftarrow \mathrm{Exp}\, x\, y \wedge m = x - 1 \wedge n + n < y \\
&\mathrm{Exp}\, m\, n \leftarrow m \leq x \wedge n \leq y \wedge \mathrm{Exp}\, x\, y
\end{aligned}$$

$$\mathit{false} \leftarrow \mathrm{Integral}\, 255\, 0\, \mathrm{Exp}$$

In the canonical interpretation, Exp represents the function defined by $f(m) = \lfloor 2^{7-m} \rfloor$ and $\mathrm{Integral}\, tot\ bd\ f$ is true if tot is less than or equal to the integral (infinite sum) of the monotone function represented by f from bd to ∞. (Thus $\max\{tot \mid$ 'Integral $tot\, 0\, \mathrm{Exp}$' holds$\} = 255$.)

This example is unsatisfiable (there is no consistent interpretation of Integral and Exp where $\mathrm{Integral}\, 255\, 0\, \mathrm{Exp}$ is false), but if the constant 255 is changed to 256, it becomes satisfiable.

Entwined structures

The key innovation of our decidability proof is the construction (given Γ) of a set of candidate models, called *entwined structures*, which satisfy a number of pleasing properties:

(P1) The set of entwined structures is r.e.
(P2) In each order-n entwined structure, the denotation of each (initial) relational type (that occurs in Γ) of order less than n is finite.
(P3) There is an algorithm that checks if a given entwined structure models Γ.
(P4) There is an entwined structure that models Γ if and only if Γ is satisfiable.

[3]Integral can equivalently be typed as $\mathbb{N} \times \mathbb{N} \to (\mathbb{N} \times \mathbb{N} \to o) \to o$.

Entwined structures are built up by induction on order, via a bootstrapping process. Their name reflects the interplay between the interpretation of terms and types during this process: the interpretation of a type of order-n (the set from which the interpretations of order-n predicate symbols are chosen) can only be given once the interpretation of the relevant predicate symbols of lower-order types has already been fixed. A family of structures $\{\mathcal{B}_n\}_{n \in \omega}$, indexed by (order) n, is *entwined*, if $\mathcal{B}_0$ is the structure on the empty signature; and in each $\mathcal{B}_n$:

- Predicate symbols in $\mathcal{B}_{n-1}$ (those of the foreground signature of order $< n$) are interpreted as per $\mathcal{B}_{n-1}$.
- Each predicate of an order-n active type $\rho = \sigma_1 \to \cdots \to \sigma_k \to o$ is interpreted as a function (in the set-theoretic $[\mathcal{B}_n[\![\sigma_1]\!] \to \cdots \to \mathcal{B}_n[\![\sigma_k]\!] \to \mathbb{B}]$) monotone in the W-typed argument.

For types $\rho = \tau \to \sigma$ of order less than n, $\mathcal{B}_n[\![\rho]\!]$ is the full function space $[\mathcal{B}_n[\![\tau]\!] \to \mathcal{B}_n[\![\sigma]\!]]$ if that is finite, otherwise it is the least collection of relational functions allowing it to support the interpretations of predicates assigned by $\mathcal{B}_{n-1}$. This results in something similar to a term model. We cannot use the term model because there can be infinitely many terms and therefore uncountably many interpretations of higher-order predicates, but our decidability proof rests on enumeration.

In an unrestricted setting, it would not make sense to interpret all the order-$(n-1)$ active predicates (i.e. predicates of active type) before interpreting the order-n predicates, because an order-$(n-1)$ active predicate may be passed an argument involving a predicate of order-n.

However, thanks to the initial type restriction, if an order-n term N of an active type has an order-m subterm M with $m > n$, then M is a subterm of some L (another subterm of N) of type σ (say) whose order is less than n. Since $\mathcal{B}_n[\![\sigma]\!]$ is finite (P2), we don't need to know all possible values of M to know all possible values of N.

We show decidability (Thm. IV.3) by exhibiting two semi-decision procedures—one for proving the existence of a model, and the other for non-existence—and running them in parallel. The former semi-decision procedure is an immediate consequence of (P1), (P3) and (P4). The latter is an application of the semi-decidability of HoCHC unsatisfiability, via a refutationally complete resolution proof system (Γ is unsatisfiable if, and only if, there is a resolution proof of $\bot$ from Γ) [7].

Outline: We begin with some technical preliminaries in Sec. II before introducing (higher-order) limit Datalog in Sec. III. We give a proof that the first-order fragment has a decidable satisfiability problem and show that satisfiability in general is undecidable. In Sec. IV we present initial restriction on types, and prove that the initial limit Datalog satisfiability problem is decidable. In Sec. V, we give examples of how first-order limit Datalog problems can be used with the background theory of tuples of naturals, and other well-quasi orderings (WQOs) with a decidable first-order theory. After a review of related work (Sec. VI), we conclude and briefly discuss some

further directions.

II. Technical preliminaries

This section introduces a restricted form of higher-order logic (Sec. II-A), higher-order constrained Horn clauses (HoCHCs) (Sec. II-B) and their proof system (Sec. II-C).

A. Relational higher-order logic

1) Syntax: For a fixed set $\mathfrak{I}$ (intuitively the types of individuals), the set of *argument types*, *relational types*, *1st-order types* and *types* (generated by $\mathfrak{I}$) are defined by mutual recursion as follows

$$\begin{array}{lrcl} \textit{Argument type} & \tau & ::= & \iota \mid \rho \\ \textit{Relational type} & \rho & ::= & o \mid \tau \to \rho \\ \textit{1st-order type} & \sigma_{\mathrm{FO}} & ::= & \iota \mid o \mid \iota \to \sigma_{\mathrm{FO}} \\ \textit{Type} & \sigma & ::= & \rho \mid \sigma_{\mathrm{FO}}, \end{array}$$

where $\iota \in \mathfrak{I}$. We sometimes abbreviate function types $\tau_1 \to \cdots \to \tau_n \to \sigma$ to $\overline{\tau} \to \sigma$. Intuitively, o (where $o \notin \mathfrak{I}$) is the type of the truth values (or Booleans). The types σ_{FO} are exactly those of the form $\bar{\iota} \to \iota$ or $\bar{\iota} \to o$, i.e. each argument is of some type $\iota_i \in \mathfrak{I}$. Moreover, each relational type has the form $\overline{\tau} \to o$. We define $\mathrm{order}(\tau_1 \to \cdots \to \tau_n \to \sigma) = n$ if σ is ι or o.

A *type environment* (typically Δ) is a function mapping *variables* (typically x, y, z) to argument types; for $x \in \mathrm{dom}(\Delta)$, we write $x : \tau \in \Delta$ to mean $\Delta(x) = \tau$. A *signature* (typically Σ, Ξ) is a set of distinct typed *symbols* $c : \sigma$, where $c \notin \mathrm{dom}(\Delta)$. A signature Σ is *1st-order* if σ is 1st-order for all $c : \sigma \in \Sigma$. We often write $c \in \Sigma$ if $c : \sigma \in \Sigma$ for some σ.

The set of Σ-*pre-terms* is given by $M ::= x \mid c \mid M\,M$ where $c \in \Sigma$. We assume that application associates to the left, and write $M\,\overline{N}$ for $M\,N_1 \cdots N_n$, assuming implicitly that M is not an application.

The typing judgement $\Delta \vdash M : \sigma$ is defined by

$$\frac{x \in \mathrm{dom}(\Delta)}{\Delta \vdash x : \Delta(x)} \quad \frac{c : \sigma \in \Sigma}{\Delta \vdash c : \sigma} \quad \frac{\Delta \vdash M_1 : \sigma_1 \to \sigma_2 \quad \Delta \vdash M_2 : \sigma_1}{\Delta \vdash M_1 M_2 : \sigma_2}$$

We say that M is a Σ-*term of type* σ if $\Delta \vdash M : \sigma$. A Σ-term is a *1st-order* Σ-*term* if the symbols in its construction are restricted to symbols $c : \sigma_{\mathrm{FO}} \in \Sigma$ and variables $x : \iota \in \Delta$.

Remark II.1. It follows from the definitions that each term $\Delta \vdash M : \bar{\iota} \to \iota$ can only contain variables of type ι_i and constants of non-relational 1st-order type (and contains no logical symbols, a similar approach is adopted in [10]).

We define a Σ-*formula* F by

$$F ::= M \mid F \vee F \mid F \wedge F \mid \neg F$$

where M is any Σ-term of type o. For a Σ-term or Σ-formula M and Σ-terms $N_1, \ldots, N_n$ and variables $x_1, \ldots, x_n$ that satisfy $\Delta \vdash N_i : \Delta(x_i)$, the *substitution* $M[N_1/x_1, \ldots, N_n/x_n]$ is defined in the standard way.

2) Semantics: There are two classic semantics for higher-order logic: *standard* and *Henkin semantics* [11]. In this paper, we will not be concerned with the latter, but the notion of frame is useful. Assume, for each $\iota \in \mathfrak{I}$, an associated set D_ι. Formally, a ***frame*** $\mathcal{F}$ assigns to each type σ a nonempty set $\mathcal{F}[\![\sigma]\!]$ such that

(i) $\mathcal{F}[\![\iota]\!] \coloneqq D_\iota$ for each $\iota \in \mathfrak{I}$
(ii) $\mathcal{F}[\![o]\!] \coloneqq \mathbb{B} \coloneqq \{0, 1\}$
(iii) For each $\sigma_1 \to \sigma_2$, $\mathcal{F}[\![\sigma_1 \to \sigma_2]\!] \subseteq [\mathcal{F}[\![\sigma_1]\!] \to \mathcal{F}[\![\sigma_2]\!]]$

where $[U \to V]$ is the set of functions from (sets) U to V.

Remark II.2. Unlike [7], we do not distinguish pre-frame and frame. Because λ-abstractions are not part of the HoCHC syntax here, the (weak) comprehension axiom in [7, p. 3] does not apply.

Example II.3 (Standard frame)**.** We define the *standard frame* $\mathcal{S}$ recursively by $\mathcal{S}[\![o]\!] \coloneqq \mathbb{B}$; $\mathcal{S}[\![\iota]\!] \coloneqq D_\iota$ for $\iota \in \mathfrak{I}$; and

$$\mathcal{S}[\![\tau \to \sigma]\!] \coloneqq [\mathcal{S}[\![\tau]\!] \to \mathcal{S}[\![\sigma]\!]]$$

Let Σ be a signature, and $\mathcal{F}$ be a frame. A $(\Sigma, \mathcal{F})$-***structure*** $\mathcal{A}$ assigns to each $c : \sigma \in \Sigma$ an element $c^{\mathcal{A}} \in \mathcal{F}[\![\sigma]\!]$ and for convenience we set $\mathcal{A}[\![\sigma]\!] \coloneqq \mathcal{F}[\![\sigma]\!]$ for types σ. A $(\Delta, \mathcal{F})$-***valuation*** α is a function such that for every $x : \tau \in \Delta$, $\alpha(x) \in \mathcal{F}[\![\tau]\!]$. For a $(\Delta, \mathcal{F})$-valuation α, variable x and $r \in \mathcal{F}[\![\Delta(x)]\!]$, $\alpha[x \mapsto r]$ is defined in the usual way.

Let $\mathcal{A}$ be a $(\Sigma, \mathcal{F})$-structure and let α be a $(\Delta, \mathcal{F})$-valuation. The *denotation* $\mathcal{A}[\![M]\!](\alpha)$ of a Σ-term M with respect to $\mathcal{A}$ and α is defined recursively by

$$\mathcal{A}[\![x]\!](\alpha) \coloneqq \alpha(x) \quad \mathcal{A}[\![c]\!](\alpha) \coloneqq c^{\mathcal{A}}$$
$$\mathcal{A}[\![M_1\,M_2]\!](\alpha) \coloneqq \mathcal{A}[\![M_1]\!](\alpha)\big(\mathcal{A}[\![M_2]\!](\alpha)\big)$$

For each term $\Delta \vdash M : \sigma$, we have $\mathcal{A}[\![M]\!](\alpha) \in \mathcal{A}[\![\sigma]\!]$. (We will write $\mathcal{A}[\![M]\!](\alpha)$ as $\mathcal{A}^{\Sigma}[\![M]\!](\alpha)$ when we need to be explicit about the signature of the Σ-terms M.)

Example II.4 (LIA)**.** In this paper, many examples will use the signature of ***linear integer arithmetic***[4] (LIA) (aka Presburger arithmetic) $\Sigma_{\mathrm{LIA}} \coloneqq \{0, 1, +, -, <, \leq, =, \neq, \geq, >\}$ and its standard model $\mathcal{A}_{\mathrm{LIA}}$.

B. Higher-order constrained Horn clauses (HoCHC)

We explicitly distinguish symbols of the *background* (bg) theory from those—in the *foreground* (fg)—which are constrained by clauses. This distinction enables a certain semantic separation required by a model construction (Def. IV.5), which is crucial to our decidability result (Thm. IV.3). [5]

Assumption 1. *Henceforth we fix a 1st-order signature* Σ_{bg}*, and a* $(\Sigma_{\mathrm{bg}}, \mathcal{S})$*-structure* $\mathcal{A}$*, and a finite signature* Σ_{fg} disjoint

[4] with the usual types $0, 1 : \iota$; $+, - : \iota \to \iota \to \iota$ and $\lhd : \iota \to \iota \to o$ for $\lhd \in \{<, \leq, =, \neq, \geq, >\}$; and we use the common abbreviation n for $\underbrace{1 + \cdots + 1}_{n}$, where $1 \leq n \in \mathbb{N}$

[5] Using notations in Def. IV.5 and Lem. IV.8, take $\leq \in \Sigma_{\mathrm{bg}}$. If $\leq^{\mathcal{A}} \in \mathcal{F}[\![\mathbb{Z} \to \mathbb{Z} \to o]\!]$ then $\mathcal{F}[\![\mathbb{Z} \to o]\!]$ must be infinite, contradicting Lem. IV.8.

from Σ_{bg} with only predicate symbols (of a relational type), typically X, Y, P and R and their variants. We will write such a pair of signatures as $\overline{\Sigma} = (\Sigma_{\mathrm{bg}}, \Sigma_{\mathrm{fg}})$.

Intuitively, Σ_{bg} and $\mathcal{A}$ correspond to the language and interpretation of the background theory, e.g. Σ_{LIA} together with its standard model $\mathcal{A}_{\mathrm{LIA}}$. In particular, *we (only) consider background theories with a single model.*

Next, we introduce higher-order constrained Horn clauses and their satisfiability problem [3].

Definition II.5. By *atom*, we mean background atom or foreground atom.

(i) A *background atom* is a 1st-order Σ_{bg}-term of type o.
(ii) A *foreground atom* is a Σ_{fg}-term of type o.

Note that a foreground atom has one of the following forms: (i) $R\,\overline{M}$ where $R \in \Sigma_{\mathrm{fg}}$, or (ii) $x\,\overline{M}$.

We use φ and A (and variants thereof) to refer to background atoms and (general) atoms, respectively.

Definition II.6 (HoCHC)**.** (i) A ***goal clause*** (typically G) is a disjunction $\neg A_1 \vee \cdots \vee \neg A_n$, where each A_i is an atom. We write $\bot$ to mean the empty (goal) clause.
(ii) If G is a goal clause, $R \in \Sigma_{\mathrm{fg}}$ and the variables in $\overline{x}$ are distinct, then $G \vee R\,\overline{x}$ is a ***definite clause***.
(iii) A ***higher-order constrained Horn clause (HoCHC)*** is a goal or definite clause.

Throughout the document, we will often write a clause $\neg A_1 \vee \cdots \vee \neg A_n \vee R\,\overline{x}$ as $R\,\overline{x} \leftarrow A_1 \wedge \cdots \wedge A_n$.

Next we give an example of HoCHCs from Sec. I, explicitly listing the types involved and illustrating the structures.

Example II.7 (A system of HoCHCs)**.** Let $\Sigma_{\mathrm{bg}} = \Sigma_{\mathrm{LIA}} \cup \{=_S\}$ and $\Sigma_{\mathrm{fg}} = \{\mathrm{Iter} : S \to \mathbb{Z} \to (S \to \mathbb{Z} \to o) \to o, \mathrm{Inc?} : S \to S \to \mathbb{Z} \to o, \mathrm{Tw} : S \to o\}$ and let Δ be a type environment satisfying $\Delta(m) = \Delta(n) = \Delta(k) = \mathbb{Z}$ and $\Delta(x) = \Delta(y) = \Delta(y') = S$ and $\Delta(p) = S \to \mathbb{Z} \to o$. The system consists of the HoCHCs in Ex. I.3, and the preceding three that define Iter.

A $\overline{\Sigma}$-*formula* is a formula where each term is either a Σ_{fg}-term or a 1st-order Σ_{bg}-term. Let $\mathcal{F}$ be a frame that agrees with the standard frame $\mathcal{S}$ on the base types $\mathfrak{I}$. Let $\mathcal{B}$ be a $(\Sigma_{\mathrm{fg}}, \mathcal{F})$-structure and let α be a $(\Delta, \mathcal{F})$-valuation. The definition of the *denotation* $\mathcal{B}[\![F]\!](\alpha)$ of a $\overline{\Sigma}$-formula F with respect to $\mathcal{B}$ and α is defined recursively by

$$\mathcal{B}[\![M]\!](\alpha) \coloneqq \begin{cases} \mathcal{A}^{\Sigma_{\mathrm{bg}}}[\![M]\!](\alpha_1) & \text{if } M \text{ a 1st-order } \Sigma_{\mathrm{bg}}\text{-term} \\ \mathcal{B}^{\Sigma_{\mathrm{fg}}}[\![M]\!](\alpha) & \text{if } M \text{ a } \Sigma_{\mathrm{fg}}\text{-term} \end{cases}$$
$$\mathcal{B}[\![F \wedge G]\!](\alpha) \coloneqq \min(\mathcal{B}[\![F]\!](\alpha), \mathcal{B}[\![G]\!](\alpha))$$
$$\mathcal{B}[\![F \vee G]\!](\alpha) \coloneqq \max(\mathcal{B}[\![F]\!](\alpha), \mathcal{B}[\![G]\!](\alpha))$$
$$\mathcal{B}[\![\neg F]\!](\alpha) \coloneqq 1 - \mathcal{B}[\![F]\!](\alpha)$$

where α_1 is taken to be some $(\Delta, \mathcal{S})$-valuation that agrees with α on the elements of Δ of type ι. The choice of such α_1 does not matter because it is only used to interpret 1st-order Σ_{bg}-formulas, which contain no variables from Δ that do not have type ι for some $\iota \in \mathfrak{I}$.

For $\overline{\Sigma}$-formulas F, we write $\mathcal{B}, \alpha \models F$ if $\mathcal{B}[\![F]\!](\alpha) = 1$, and $\mathcal{B} \models F$ if $\mathcal{B}, \alpha' \models F$ for all α'. We extend $\models$ in the usual way to sets of formulas.

Definition II.8. Let Γ be a set of HoCHCs, and suppose $\mathcal{F}$ is a frame which agrees with $\mathcal{S}$ on $\mathfrak{I}$.

(i) Γ is $(\mathcal{A}, \mathcal{F})$-***satisfiable*** if there exists a $(\Sigma_{\mathrm{fg}}, \mathcal{F})$-structure $\mathcal{B}$ such that $\mathcal{B} \models \Gamma$.
(ii) Γ is $\mathcal{A}$-***satisfiable*** (also called $\mathcal{A}$-*standard-satisfiable*) if it is $(\mathcal{A}, \mathcal{S})$-satisfiable.

Whilst the notion of $(\mathcal{A}, \mathcal{F})$-satisfiability may seem obscure, it is sometimes easier to construct $(\Sigma_{\mathrm{fg}}, \mathcal{F})$-structures (cf. Lem. IV.11); and for certain $\mathcal{F}$, $\mathcal{A}$-satisfiability implies $(\mathcal{A}, \mathcal{F})$-satisfiability (cf. Lem. IV.12).

Definition II.9. A ***program*** is a finite set of definite clauses.

Remark II.10. (i) Under the definition of satisfiability, a program can be seen as a conjunction of clauses, universally quantified over variables in Δ.

(ii) It is often convenient to write logically equivalent formulas such as $X\,x \leftarrow \exists y. Y\,x\,y \vee Z\,x$ instead of $\{X\,x \vee \neg Y\,x\,y,\ X\,z \vee \neg Z\,z\}$. We may even write the bodies of such formulas as existentially quantified formulas, over variables that do not appear in the head.

(iii) If Σ_{bg} contains a predicate interpreted by $\mathcal{A}$ as equality (as with Σ_{LIA} and $\mathcal{A}_{\mathrm{LIA}}$), then we may write terms of integer type inside foreground atoms. For example $X\,(x + y + 5)$ is equivalent to $X\,z \wedge (z = x + y + 5)$

(iv) Every satisfiable set of clauses Γ has, in each frame, a ***canonical model*** $\mathcal{B}$ which arises by saturating under all immediate consequences [7, Thm. 23]. In the higher-order setting, this model may not be least wrt inclusion, but for any goal clause G we have: $\Gamma \cup \{G\}$ satisfiable iff $\mathcal{B} \models G$.

C. Resolution proof system

We use a simple resolution proof system [7] consisting of only two rules: (i) a higher-order version of the usual resolution rule [12] between a goal clause and a definite clause (thus yielding a goal clause) and (ii) a rule to refute certain goal clauses which are not satisfied by the model of the background theory (similar to [13]).

$$\textbf{Resolution} \quad \frac{\neg R\,\overline{M} \vee G \qquad G' \vee R\,\overline{x}}{G \vee \left(G'[\overline{M}/\overline{x}]\right)}$$

$$\textbf{Refutation} \quad \frac{\neg x_1\,\overline{M}_1 \vee \cdots \vee \neg x_m\,\overline{M}_m \vee \neg\varphi_1 \vee \cdots \vee \neg\varphi_n}{\bot}$$

With the latter rule applicable only there exists a valuation α such that $\mathcal{A}, \alpha \models \varphi_1 \wedge \cdots \wedge \varphi_n$. Since variables are implicitly universally quantified, we may assume $x_1 \cdots x_m$ are interpreted as $(\bar{y} \mapsto \mathsf{false})$. The rules must be applied modulo

renaming of (free) variables; we write $\Gamma' \vdash_{\mathcal{A}} \Gamma' \cup \{G\}$ if G can be thus derived from the clauses in Γ' using the above rules and $\vdash^*_{\mathcal{A}}$ for the reflexive, transitive closure of $\vdash_{\mathcal{A}}$.

Theorem II.11 (Soundness and Completeness [7])**.** *Let Γ be a set of HoCHCs. Then Γ is $\mathcal{A}$-unsatisfiable if, and only if, $\Gamma \vdash^*_{\mathcal{A}} \{\bot\} \cup \Gamma'$ for some Γ'.*

It follows that a set of HoCHCs is $\mathcal{A}$-satisfiable if, and only if, it cannot be refuted by the proof system.

Consequently, the resolution proof system gives rise to a semi-decision procedure for the (standard) $\mathcal{A}$-unsatisfiability problem provided the consistency[6] of conjunctions of atoms in the background theory is semi-decidable.

III. Higher-order limit Datalog

In this section, we describe the limit restriction on HoCHC programs. We discuss the first-order fragment with this restriction, showing in Sec. III-A that its satisfiability problem is decidable. Then, in Sec. III-B, we show that this does not hold for higher-order problems, motivating the restrictions described in the rest of this paper.

To begin, we need some properties of the background theory, so we extend Assumption 1 by

Assumption 2. *Henceforth fix some set W, a 1st-order signature Σ_W and a $(\Sigma_W, \mathcal{S})$-structure $\mathcal{A}_W$ such that the first-order theory of $(\Sigma_W, \mathcal{A}_W)$ is decidable, $\leq \in \Sigma_W$, $\leq^{\mathcal{A}_W}$ is a preorder on W, and for each upset X (i.e. a subset of W such that if $x \in X$ and $x \leq^{\mathcal{A}_W} y$ then $y \in X$), there is a Σ_W-formula $\varphi(x)$ which expresses membership of X.*

Moreover fix some finite set S. We strengthen Assumption 1 by asserting that:

- $\Sigma_{\mathrm{bg}} := \Sigma_W \cup \{=_S : S \to S \to o\} \cup \{s : S \mid s \in S\}$
- $c^{\mathcal{A}} := c^{\mathcal{A}_W}$ *if* $c : \sigma \in \Sigma_W$*;* $s^{\mathcal{A}} := s$ *if* $s \in S$*; and* $(=_S)^{\mathcal{A}}$ *is the standard equality between elements of S.* □

Note that the constraints on W imply that there are countably many upsets. Examples of such structures include the integers with $\mathcal{A}_{\mathrm{LIA}}$ (the upsets are either $\mathbb{Z}$, $\emptyset$ or $\{x : x \geq k\}$ for some $k \in \mathbb{Z}$, each of which can easily be described by a formula) and any countable well-quasi-ordering (WQO) with a decidable background theory (for example, tuples of naturals under component-wise ordering also with the theory of linear integer arithmetic). Also note that any predicate on S can be expressed in terms of $=_S$, so our examples may freely make use of other predicates.

Recall that a *well-quasi-ordering* (WQO) [14] is a quasi-order $(W, \leq)$ such that every infinite sequence $w_1, w_2, \cdots$ contains an increasing pair: $w_i \leq w_j$ for some $i < j$. To see that all upsets of a countable WQO $(W, \leq)$ are expressible, note that any upset $X \subseteq W$ has a finite number of minimal elements $m_1, \cdots, m_n$ (say), hence X can be described as $\{w \in W : \bigvee_{i=1}^{n} w \geq m_i\}$. Unfortunately, this does mean that all elements of W must be constants in background theory, which makes it harder to obtain decidability (for example, the subword order is a WQO, but with constants, even the $\exists$-theory becomes undecidable [8, 15]).

Remark III.1. Note that the converse of a preorder is another preorder, and upsets under one are downsets (the complements of upsets) under the other. This means that if Assumption 2 holds for a relation, it also holds for its converse. We will make use of this by using upwards closed predicates in the definition below and in the proofs for technical convenience, despite the examples in Sec. I using downwards closed predicates.

Definition III.2. (i) An ***(upwards) limit Datalog problem*** Γ is a finite set of HoCHC clauses over a signature $\overline{\Sigma} = (\Sigma_{\mathrm{bg}}, \Sigma_{\mathrm{fg}})$, compatible with Assumptions 1 and 2, such that for each $X : \rho \in \Sigma_{\mathrm{fg}}$, ρ contains at most one argument of type W, and if $\rho = \tau_1 \to \ldots \to \tau_n \to W \to \tau_{n+1} \to \ldots \tau_{n+m} \to o$ then Γ contains a *limit clause*:

$$X\,\overline{z}\,x\,\overline{z'} \;\leftarrow\; y \leq x \wedge X\,\overline{z}\,y\,\overline{z'}$$

writing $\overline{z}$ and $\overline{z'}$ for $z_1 \ldots z_n$ and $z_{n+1} \ldots z_{n+m}$ respectively.

(ii) A ***first-order limit Datalog problem*** Γ is a limit Datalog problem where for each $X : \rho \in \Sigma_{\mathrm{fg}}$, it is the case that $\rho \in \sigma_{\mathrm{FO}}$, and no atom that occurs in Γ is headed by a variable.

(iii) The ***satisfiability problem*** for limit Datalog asks: given a limit Datalog problem Γ, is it $\mathcal{A}$-satisfiable?.

A key idea of limit Datalog is that predicates with W-typed arguments must be interpreted as sets that are closed upward with respect to that argument. Consequently, a proposition $X\,\overline{z}\,y\,\overline{z'}$ asserts only that X holds of "at least y" (i.e., X is a min-predicate in the sense of [6]).

First-order limit Datalog$_{\mathbf{Z}}$: [6] describe *first-order limit Datalog$_{\mathbf{Z}}$* which is first-order limit Datalog over linear integer arithmetic. Examples of this (Examples I.1 and I.2) are given in Sec. I.

Remark III.3. [6] also allow predicates defining finite sets of integers; and both min- and max-predicates; and multiplication by constants, and by integers from fixed finite sets. They show that a limit Datalog$_{\mathbf{Z}}$ problem with these features can be transformed into one without them.

First-order limit Datalog$_{\mathbf{Z}}$ is motivated by aggregation in declarative data analysis, which is typified by its requirements for recursion and linear integer arithmetic. In declarative data analysis, the emphasis is on giving a specification of the required output rather than instructions on how to achieve it. Such an analysis is enabled by a declarative language, and experience suggests that support for high-level programming over collection types (e.g. list comprehensions, map, reduce) is particularly beneficial [16]. Consequently, a higher-order foundation, such as HoCHC, may be particularly appropriate.

Of course higher-order programming is most important for larger codebases where it can be reused many times, but Examples I.3 and I.4 show that already the (first-order)

[6] i.e. whether there exists a valuation α such that $\mathcal{A}, \alpha \models \varphi_1 \wedge \cdots \wedge \varphi_n$

examples given in [6] have a shared structure that can be factored out using a higher-order combinator.

A. Decidability at order 1

A key result of [6] is that the decision problem for the first-order language is decidable. We give an alternative proof of this theorem extended to first-order limit Datalog (allowing for structures other than linear integer arithmetic) which is helpful when understanding similar proofs in the sections that follow.

Theorem III.4. *The satisfiability problem for first-order (upwards) limit Datalog is decidable.*

Proof. Take a first-order limit Datalog problem Γ. It follows from Def. III.2 that for any $(\Sigma_{\text{fg}}, \mathcal{S})$-structure $\mathcal{B}$ such that $\mathcal{B} \models \Gamma$, predicate $X : S^n \to W \to S^m \to o \in \Sigma_{\text{fg}}$, and tuples of constants $\overline{s}, \overline{s'} \subseteq S$, the set

$$U = \{w \in W \mid \mathcal{B}[\![X\,\overline{s}\,x\,\overline{s'}]\!]([x \mapsto w]) = 1\}$$

is upwards closed. By Assumption 2, there exists a 1st-order formula $\varphi_{X,\overline{s},\overline{s'}}(x)$ such that

$$U = \{w \in W \mid \mathcal{A}_W[\![\varphi_{X,\overline{s},\overline{s'}}(x)]\!]([x \mapsto w])\}.$$

As there are finitely many predicates, finitely many tuples of elements of S and countably many such formulas φ, we have an r.e. set of candidate models. Given a $(\Sigma_{\text{fg}}, \mathcal{S})$-structure $\mathcal{B}$ of this form, we may ground all instances of variables from S, then substitute formulas $\varphi_{X,\overline{s},\overline{s'}}$ as appropriate, removing all instances of predicate symbols. Since the 1st-order theory of $(W, \mathcal{A}_W)$ is decidable, we can decide if $\mathcal{B} \models \Gamma$.

If Γ is satisfiable, we can find such a structure by enumeration. If not, then resolution (Thm. II.11) can prove that. □

Corollary III.5. *The satisfiability problem for first-order downwards limit Datalog is decidable.*

Proof. Although the above proof covers upwards limit Datalog, it only relies on the fact that upwards closed sets are expressible as 1st-order formulas. Since the complement of every downwards closed set D is an upwards closed set U, D is described by the negation of the formula describing U. Thus the proof also holds for downwards limit Datalog. □

B. Undecidability in general

Unlike the first-order case, higher-order limit Datalog in general is undecidable[7], which can be proved by demonstrating that multiplication, hence Diophantine equations, is definable.

The idea is to use a pair of terms of type $\mathbb{Z} \to o$ to represent an integer. Fix a higher-order limit Datalog$_{\mathbf{Z}}$ program Γ and let $\mathcal{B}$ be its canonical model. For an integer k, we write $[\![(M, N)]\!] \equiv k$ just if $\mathcal{B}[\![M]\!] = \{x : x \geq k\}$ and $\mathcal{B}[\![N]\!] = \{x : x \geq -k\}$. This ensures that, for any $n \in \mathbb{Z}$, $\mathcal{B}[\![M\,n \wedge N\,(-n)]\!] = 1$ iff $n = k$. Then we say that a partial function $f : \mathbb{Z}^m \to \mathbb{Z}$ is *definable in* Γ just if there exist two closed terms M_1 and M_2 of type

$$\underbrace{(\mathbb{Z} \to o) \to \cdots \to (\mathbb{Z} \to o)}_{2m\text{-times}} \to (\mathbb{Z} \to o)$$

such that: if for each $i \in \{1, \ldots, m\}$, $[\![(P_i, P_i')]\!] \equiv k_i$ then

$$\begin{aligned} & [\![(M_1\,P_1\,P_1' \cdots P_m\,P_m', M_2\,P_1\,P_1' \cdots P_m\,P_m')]\!] \\ \equiv \quad & f(k_1, \ldots, k_m). \end{aligned}$$

Example III.6 (Addition)**.** Consider the following program which defines addition and the constant 5.

$$\begin{gathered} \mathrm{Add}_1, \mathrm{Add}_2 : \sigma \to \sigma \to \sigma \to \sigma \to \mathbb{Z} \to o \\ \mathrm{I}_{51}, \mathrm{I}_{52} : \sigma \quad \text{where } \sigma = \mathbb{Z} \to o \end{gathered}$$

$$\begin{aligned} &\mathrm{Add}_1\,f_1\,f_2\,g_1\,g_2\,x \\ &\qquad \leftarrow f_1\,y \wedge f_2\,(-y) \wedge g_1\,z \wedge g_2\,(-z) \wedge x \geq y + z \\ &\mathrm{Add}_2\,f_1\,f_2\,g_1\,g_2\,x \\ &\qquad \leftarrow f_1\,y \wedge f_2\,(-y) \wedge g_1\,z \wedge g_2\,(-z) \wedge x \geq -(y + z) \\ &\mathrm{I}_{51}\,x \leftarrow x \geq 5 \\ &\mathrm{I}_{52}\,x \leftarrow x \geq -5 \end{aligned}$$

In the canonical model of this program,

$$\mathrm{Add}_1\,\mathrm{I}_{51}\,\mathrm{I}_{52}\,\mathrm{I}_{51}\,\mathrm{I}_{52}\,x \wedge \mathrm{Add}_2\,\mathrm{I}_{51}\,\mathrm{I}_{52}\,\mathrm{I}_{51}\,\mathrm{I}_{52}\,(-x)$$

would hold exactly when $x = 10$. This means that the pair of partially applied functions $\mathrm{Add}_1\,\mathrm{I}_{51}\,\mathrm{I}_{52}\,\mathrm{I}_{51}\,\mathrm{I}_{52}$ and $\mathrm{Add}_2\,\mathrm{I}_{51}\,\mathrm{I}_{52}\,\mathrm{I}_{51}\,\mathrm{I}_{52}$ can be used as arguments to other functions; for example

$$\mathrm{Add}_1\,(\mathrm{Add}_1\,\mathrm{I}_{51}\,\mathrm{I}_{52}\,\mathrm{I}_{51}\,\mathrm{I}_{52})\,(\mathrm{Add}_2\,\mathrm{I}_{51}\,\mathrm{I}_{52}\,\mathrm{I}_{51}\,\mathrm{I}_{52})\,\mathrm{I}_{51}\,\mathrm{I}_{52}\,x$$

would hold for $x \geq 15$.

In [**?**], we give another example of how functions may be composed, and recursion can work, by defining multiplication. With this we can define a goal clause corresponding to any Diophantine equation, in such a way that the program as a whole is satisfiable iff the equation has a solution. Consequently:

Theorem III.7 (Undecidability)**.** *The satisfiability problem for higher-order limit Datalog$_{\mathbf{Z}}$ is undecidable.*

Proof. Since solvability of Diophantine equations is undecidable [**?**], so is the problem of determining if a higher-order limit Datalog$_{\mathbf{Z}}$ problem is satisfiable. □

IV. Initial limit Datalog

In this section, we prove Thm. IV.3, which says that a particular fragment of higher-order limit Datalog is decidable. The proof follows the same strategy as that of Thm. III.4. The key difference occurs when we enumerate candidate models; even though we can restrict the first-order predicates to an enumerable set, there are still uncountably many inhabitants of higher-order types under standard semantics.

[7]The proof given here covers integers with linear integer arithmetic. A variant works for naturals, but higher-order limit Datalog is not undecidable for all structures $(W, \mathcal{A}_W)$.

To work around this, first note that there are finitely many predicate symbols. If these were the only higher-order terms, we would be fine since Thm. II.11 can be seen as saying that satisfiability does not depend on the behaviour of predicates on elements of higher-order function spaces that don't correspond to terms. However, terms can contain arbitrarily deeply nested subterms, as seen in Ex. III.6 (and used in the proof of undecidability). This means there can be a countable infinity of terms with distinct interpretations, leading to an uncountable infinity of interpretations for predicates over those terms. We can prevent this kind of nesting by restricting the types of predicates in the following way.

We insist that among the arguments to a predicate from Σ_{fg}, at most one is of type W, and every argument that occurs to the left of the W-typed argument (if there is one) must be of a smaller order than this function of W. For example, we would admit predicates of type $S \to W \to o$ and $(W \to o) \to W \to S \to (W \to o) \to o$, but not those of type $W \to W \to o$ nor $(W \to o) \to W \to o$.

Definition IV.1. (i) An ***initial*** type is a relational type $\sigma_1 \to \cdots \to \sigma_n \to o$ where $n \geq 0$ satisfying
- (O1) at most one of $\sigma_1, \cdots, \sigma_n$ is W, and
- (O2) if $\sigma_j = W$ then for all $i < j$, $\text{order}(\sigma_i) < \text{order}(\sigma_j \to \sigma_{j+1} \to \cdots \to \sigma_n \to o)$, and
- (O3) each σ_j is S, or W, or initial.

(ii) Let $\rho = \overline{\sigma} \to o$ be an initial type. We say that ρ is an ***active*** type (typically ξ) if some σ_i is W; otherwise it is an ***inactive*** type (typically ν).

(iii) An ***initial limit Datalog problem*** is a limit Datalog problem where for each $X : \rho \in \Sigma_{\text{fg}}$, ρ is initial.

Example IV.2. All the types in Examples I.3 to I.6 are initial; but neither Add_1 nor Add_2 in Ex. III.6 have an initial type.

Theorem IV.3 (Decidability)**.** *Given Assumptions 1 and 2, there is an algorithm that decides whether a given initial limit Datalog problem is $\mathcal{A}$-satisfiable.*

The initial type restriction does not prevent nested terms, but it does prevent problematic ones by making the subterm relationship compatible with the type-theoretic order of the terms involved. If an order-n term N of an active type contains an order-m subterm M where $m > n$, then M is a subterm of some L (another subterm of N) of type σ (say) whose order is less than n. (This is because N must have the form $X\,\overline{L}$ where $X \in \Sigma_{\text{fg}}$ is an active type, and each L_i has order less than n.) We will see later that we can take the interpretation of this type σ to be a finite set, and hence we don't need to know all possible values of M to know all possible ways in which we can interpret N.

This ensures that we can enumerate candidate models up to their behaviour on constructible elements. It allows (a) for interpretations to be defined inductively: the interpretation of all order-n predicates is given before any of order-$(n+1)$ and (b) the behaviour of a predicate on a non-W argument need only be specified on *finitely many* definable elements (Lem. IV.8).

Consider a predicate symbol $X : (W \to o) \to o \in \Sigma_{\text{fg}}$. Without restriction, there may be an infinity of definable elements of type $W \to o$ and hence uncountably many choices of interpretation of X. However, a Σ_{fg}-term of type $W \to o$ can only be constructed by applying a predicate symbol Y to some arguments $N_1, \cdots, N_k$. It follows that Y has a type of shape $\sigma_1 \to \cdots \to \sigma_k \to W \to o$. By the initial type restriction, each σ_i is necessarily S and hence finite. If we have already fixed the interpretation of each such Y (each being of lower order than X), then there are only finitely many definable elements at type $W \to o$. Hence, there are only finitely many definable relations at type $(W \to o) \to o$.

Of course, when first fixing the interpretation of Y there can be infinitely many choices; but thanks to the limit Datalog restriction, only countably many can satisfy the limit clause which requires that any such interpretation is upward-closed in its W argument. It is straightforward to see that the choices are, moreover, r.e. (Lem. IV.10).

This leads to the notion of an interpretation that is built up inductively by order, in which the domains of the higher-order predicates (i.e. interpretation of types) are not determined until the interpretations of lower-order predicates have been fixed. The process of choosing interpretations for the types (i.e. the frame) and the process of choosing interpretations for the predicate symbols are entwined.

Assumptions: Recall disjoint signatures Σ_{bg} and Σ_{fg} and 1st-order structure $\mathcal{A}$ from Assumptions 1 and 2. Henceforth fix an initial limit Datalog problem Γ and take

$$l := \max\{\text{order}(\rho) \mid X : \rho \in \Sigma_{\text{fg}}\}.$$

Definition IV.4. Let Ξ_1 and Ξ_2 be (possibly higher-order) signatures such that $\Xi_1 \subseteq \Xi_2$; and $\mathcal{F}_1$ and $\mathcal{F}_2$ be frames.

Suppose $\mathcal{B}_1$ is a $(\Xi_1, \mathcal{F}_1)$-structure. We say that a $(\Xi_2, \mathcal{F}_2)$-structure $\mathcal{B}_2$ is a $(\Xi_2, \mathcal{F}_2)$-***expansion*** of $\mathcal{B}_1$ just if $c^{\mathcal{B}_2} = c^{\mathcal{B}_1}$ for all $c \in \Xi_1$.

We first define, given sets $U_1, \cdots, U_n$, a relation on relational functions, $\leq_{o,n} \subseteq [U_n \to \cdots \to U_1 \to \mathbb{B}]^2$, by

$$f \leq_{o,0} g \coloneqq (f = 0 \text{ or } g = 1)$$
$$f \leq_{o,n+1} g \coloneqq \forall x \in U_{n+1} \,.\, f(x) \leq_{o,n} g(x)$$

Define $\top_n \in [U_n \to \cdots \to U_1 \to \mathbb{B}]$ as $\forall \overline{x} \,.\, \top_n \overline{x} = 1$. Henceforth we elide the subscript n from $\leq_{o,n}$ and $\top_n$.

Definition IV.5. Let $n \geq 1$ and $\Xi \subseteq \Sigma_{\text{fg}}$. Given a $(\Xi, \mathcal{F})$-structure $\mathcal{B}$, define the ***entwined order-n frame derived from*** $\mathcal{B}$, written $\langle \mathcal{B} \rangle_n$, by case analysis of σ as follows.

(i) σ is initial and $\text{order}(\sigma) \leq n-2$, or σ is (inactive, or S, or W, or o) and $\text{order}(\sigma) \leq n-1$:

$$\langle \mathcal{B} \rangle_n [\![\sigma]\!] := \mathcal{F}[\![\sigma]\!]$$

(ii) σ active and order$(\sigma) = n-1$:

$$\langle\mathcal{B}\rangle_n[\![W \to \nu]\!] := \{\top \in [W \to \langle\mathcal{B}\rangle_n[\![\nu]\!]]\} \cup \{X^{\mathcal{B}}\,\overline{s} \mid X : \overline{\tau} \to W \to \nu \in \Xi,\ s_i \in \mathcal{F}[\![\tau_i]\!]\}$$
$$\langle\mathcal{B}\rangle_n[\![\sigma_1 \to \xi]\!] := [\langle\mathcal{B}\rangle_n[\![\sigma_1]\!] \to \langle\mathcal{B}\rangle_n[\![\xi]\!]]$$

(iii) σ is initial and order$(\sigma) = n$:

$$\langle\mathcal{B}\rangle_n[\![W \to \nu]\!] := \{f \in [W \to \langle\mathcal{B}\rangle_n[\![\nu]\!]] \mid \forall z \leq^{\mathcal{A}} z'.f(z) \leq_o f(z')\}$$
$$\langle\mathcal{B}\rangle_n[\![\sigma_1 \to \sigma_2]\!] := [\langle\mathcal{B}\rangle_n[\![\sigma_1]\!] \to \langle\mathcal{B}\rangle_n[\![\sigma_2]\!]] \quad (\sigma_1 \neq W)$$

(iv) σ is not initial, or order$(\sigma) > n$:

$$\langle\mathcal{B}\rangle_n[\![\sigma_1 \to \sigma_2]\!] := [\langle\mathcal{B}\rangle_n[\![\sigma_1]\!] \to \langle\mathcal{B}\rangle_n[\![\sigma_2]\!]]$$

The preceding definition is used in the context of entwined structures (Def. IV.6).

In that context, we explain the cases: By Lem. IV.8, sorts covered by case (i) can be treated as finite, so are easy to deal with. Case (ii) is the most interesting - it is where we make use of the structure $\mathcal{B}$. Here we set the interpretation of order-$(n-1)$ active types $W \to \nu$ to be the minimum ensuring that $\mathcal{B}$ is still a $\langle\mathcal{B}\rangle_n$-structure. Top ($\top$) is needed for technical reasons. Case (iii) keeps things countable by discarding interpretations that don't satisfy the limit clauses. This is exactly like the proof of Thm. III.4. This defines the space from which we will pick interpretations for order-n predicate symbols (say $\mathcal{B}'$), and $\langle\mathcal{B}'\rangle_{n+1}$ will fix the space to become finite. Case (iv) of Def. IV.5 is only there to ensure that $\langle\mathcal{B}\rangle_n$ is technically a frame. Such types are not used anywhere.

For $i \geq 1$, let $\Sigma_i \subseteq \Sigma_{\text{fg}}$ consist of the predicate symbols of Σ_{fg} with types of order at most i.

Definition IV.6. (i) A family of structures $\{\mathcal{B}_n\}_{n\in\omega}$, indexed by (order) n, is said to be ***entwined*** just if $\mathcal{B}_0$ is the unique $(\emptyset, \mathcal{S})$-structure, and each $\mathcal{B}_{n+1}$ is a $(\Sigma_{n+1}, \langle\mathcal{B}_n\rangle_{n+1})$-expansion of $\mathcal{B}_n$.

(ii) An ***entwined structure*** is a member of some entwined family. An *entwined model* of Γ is an entwined structure $\mathcal{B}_{l+1}$ such that $\mathcal{B}_{l+1} \models \Gamma$.

Example IV.7. Using the background theory LIA (so $W = \mathbb{Z}$), take, for example, the term $X\,a\,(Y\,b)\,(Z\,5\,X)$ for some a, b such that $\Delta(a), \Delta(b) = S$ and

$$Y : S \to S \to o$$
$$X : \rho = S \to (S \to o) \to o \to W \to (W \to o) \to o$$
$$Z : W \to \rho \to o$$

Now Z has a complicated type, but $Z\,5\,X$ must be either true or false, so we can select behaviours for X ignorant of Z (and the choices for Z can depend on this without introducing a problematic cycle). This example is elaborated in [**?**].

In the following lemmas (full proofs for which are available in [**?**]), let $\{\mathcal{B}_n\}_{n\in\omega}$ be an entwined family, and set $\mathcal{F}_n := \langle\mathcal{B}_{n-1}\rangle_n$. Note that $\Sigma_{l+1} = \Sigma_{\text{fg}}$.

Lemma IV.8. *Let σ be an initial type. If $n >$ order(σ), or $n =$ order(σ) and σ is an inactive type, then $\mathcal{F}_n[\![\sigma]\!]$ is finite.*

Lemma IV.9. *Let σ be an initial active type. If $n =$ order(σ) then $\mathcal{F}_n[\![\sigma]\!]$ is r.e.*

Lemma IV.10. *The set of entwined families of structures is r.e.*

For each resolution proof rule, if $\mathcal{B}_{l+1}$ entails the premises of the rule, then it entails the conclusion. Since $\mathcal{B}_{l+1}[\![\bot]\!] = 0$, there is no resolution proof of $\bot$.

Lemma IV.11. *If there is an entwined family such that $\mathcal{B}_{l+1}$ models Γ, then there is no resolution proof of $\bot$ from Γ.*

On the other hand, the inductive construction gives enough freedom to choose appropriate interpretations for the predicate symbols whenever the clauses are satisfiable. Any model can be reconstructed as an entwined structure that also satisfies the clauses, with the relationship between the two mediated by a logical relation.

Lemma IV.12. *If Γ is satisfiable then there is an entwined structure that models Γ.*

Observe that, for any entwined family, $\mathcal{F}_{l+1}[\![\rho]\!]$ is finite whenever $X : \rho \in \Sigma_{\text{fg}}$. We can ask whether a particular $\mathcal{B}_{l+1} \models \Gamma$ and this is decidable because it is equivalent to a formula in the first-order theory of Σ_W.

Lemma IV.13. *Given an entwined structure $\mathcal{B}_{l+1}$, determining if it satisfies a goal or definite clause G is decidable.*

Proof of Thm. IV.3: If there is a refutation of Γ by resolution, then we know Γ is $\mathcal{A}$-unsatisfiable. By Lem. IV.11, there is no entwined structure $\mathcal{B}_{l+1}$ such that $\mathcal{B}_{l+1} \models \Gamma$.

If there is no resolution proof of $\bot$, then there is some model for Γ in standard semantics. This model can be converted into an entwined model by Lem. IV.12. Hence enumerating entwined structures—possible because they are r.e. (Lem. IV.10) and determining if $\mathcal{B}_{l+1} \models \Gamma$ is decidable (Lem. IV.13)—will find a model.

Therefore, we may interleave a search for resolution proofs of $\bot$ with a search for entwined models resulting in a decision procedure for the initial limit Datalog decision problem. □

Example IV.14. For a concrete example of an entwined structure, see [**?**].

Remark IV.15 (Higher-order initial limit Datalog$_{\mathbf{Z}}$). It can be shown that higher-order initial limit Datalog$_{\mathbf{Z}}$ is strictly more expressive than first-order limit Datalog$_{\mathbf{Z}}$. By this, we mean that there are queries about databases (aka structures on finite sets) that can be expressed with higher-order initial limit Datalog$_{\mathbf{Z}}$ but not first-order limit Datalog$_{\mathbf{Z}}$. This follows from a result in [17] which shows that the data complexity of k-order Datalog lies in $(k-1)$-EXPTIME. Since this only uses finite sets, the programs involved are valid higher-order

initial limit Datalog$_\mathbf{Z}$ programs. [6] shows that first-order limit Datalog$_\mathbf{Z}$ has more reasonable time bounds (coNP-complete in database size) hence it must be less expressive.

V. Examples

In this section, we give examples of how first-order limit Datalog problems can be used with the background theory of tuples of naturals (we use currying to avoid explicitly specifying projection functions), and other WQOs.

A. Theory of tuples of naturals

In the context of limit Datalog, the theory of tuples of naturals with componentwise ordering is much more powerful than that of integers as demonstrated by the examples below, the latter of which could not be accomplished using limit Datalog$_\mathbf{Z}$.

The following set of clauses over $F : S \to \mathbb{N} \to \mathbb{N} \to o$ and $G : \mathbb{N} \to o$ express multiplication of pairs of data items in a database:

$$\begin{aligned} F\,s\,x\,y &\leftarrow x \geq 0 \wedge y \geq a \wedge D(s,a,b) \\ F\,s\,x\,y &\leftarrow y+1 \geq n \wedge F\,r\,n \wedge x \geq r+b \wedge D(s,a,b) \\ G\,x &\leftarrow F\,s\,x\,0 \end{aligned}$$

Here D is a database predicate and we assume that for each $s \in S$ there is a unique pair of natural numbers a, b such that $(s,a,b) \in D$. In the canonical model of this set of clauses, the interpretation of G is the set $\{x \in \mathbb{N} : \exists a,b,s.(s,a,b) \in D \wedge x \geq ab\}$.

This may be extended to express exponentiation α^β (demonstrated below) and further to other hyperoperations.

$$\begin{aligned} &F : \mathbb{N} \to \mathbb{N} \to \mathbb{N} \to o \\ F\,x\,y\,z &\leftarrow x \geq 0 \wedge y \geq \alpha \wedge z \geq 0 \\ F\,x\,y\,z &\leftarrow x \geq 1 \wedge y \geq 0 \wedge z \geq \beta \\ F\,x\,y\,z &\leftarrow z+1 \geq n \wedge F\,d\,0\,n \wedge y+1 \geq m \wedge \\ &\qquad F\,r\,m\,z \wedge x \geq r+d \\ G\,x &\leftarrow F\,x\,0\,0 \end{aligned}$$

B. Lossy counter machines

A *(classic) lossy n-counter machine (n-LCM)*, due to [18], consists of: a finite set of states Q, an initial state $q_0 \in Q$, a final state $q_f \in Q$, n counters $c_1, \ldots, c_n$, and a finite set of instructions, each of one of the two shapes A or B:

A. $(q:\ c_i := c_i + 1;\ \mathsf{goto}\ q')$
B. $(q:\ \mathsf{if}\ c_i = 0\ \mathsf{then\ goto}\ q'\ \mathsf{else}\ c_i := c_i - 1;\ \mathsf{goto}\ q'')$

A *configuration* s of such a machine is an $(n+1)$-tuple of shape $(q, m_1, \ldots, m_n)$ where $q \in Q$ and each $m_i \in \mathbb{N}$ being the current value of counter c_i.

A *transition* of such a machine consists of spontaneous loss, followed by the execution of an instruction, followed by spontaneous loss:

$$s_1 \Rightarrow s_2 \quad \text{iff} \quad \exists s_1', s_2'.\ s_1 \xrightarrow{l} s_1' \to s_2' \xrightarrow{l} s_2$$

The execution of an instruction $(p, m_1, \ldots, m_i, \ldots, m_n) \to (p', m_1, \ldots, m_i', \ldots, m_n)$ is defined iff:

- there is an instruction of shape A and $p = q$, $m_i' = m_i + 1$ and $p' = q'$
- or, there is an instruction of shape B and $p = q$, $m_i = 0$, $m_i' = 0$ and $p' = q'$
- or, there is an instruction of shape B and $p = q$, $m_i > 0$, $m_i' = m_i - 1$ and $p' = q''$.

The spontaneous (classic) loss $(q, m_1, \ldots, m_n) \xrightarrow{l} (q, m_1', \ldots, m_n')$ is defined iff $\forall i \in [1,n].\ m_i' \leq m_i$. Let us write $\Rightarrow^*$ for the reflexive, transitive closure of the transition relation.

The *reachability problem* for n-LCM is to decide the following: given a configuration s, does $(q_0, 0, \ldots, 0) \Rightarrow^* s$? It is known that the reachability problem is decidable as a special case of [19]. Here we give an alternative approach using initial limit Datalog over tuples of natural numbers.

To decide the problem, it suffices to construct a set of definite clauses C over the foreground signature

$$\{R_q : \mathbb{N} \to \cdots \to \mathbb{N} \to o \mid q \in Q\}$$

(each R_q is of arity n) with canonical model $\mathcal{B}$, in such a way that, for each state q, we have $\mathcal{B} \models R_q\ m_1\ \cdots m_n$ iff $(q_0, 0, \ldots, 0) \Rightarrow^* (q, m_1, \ldots, m_n)$. We define C as follows, abbreviating $x_1 \cdots x_n$ and $y_1 \cdots y_n$ by $\vec{x}$ and $\vec{y}$ respectively.

- The clause $R_{q_0}\,\vec{x} \leftarrow \bigwedge_{i\in[1,n]} x_i = 0$ is in C.
- For each state $q \in Q$, the following limit clause is in C:

$$R_q\ \vec{x} \leftarrow R_q\ \vec{y} \wedge \bigwedge_{i\in[1,n]} x_i \leq y_i$$

- For each instruction of shape A, the following clause:

$$R_{q'}\ \vec{x} \leftarrow R_q\ \vec{y} \wedge x_i = y_i + 1 \wedge \bigwedge_{j\in[1,n]\setminus\{i\}} x_j = y_j$$

- For each instruction of shape B, the two clauses:

$$R_{q'}\ \vec{x} \leftarrow R_q\ \vec{y} \wedge y_i = 0 \wedge \bigwedge_{j\in[1,n]} x_j = y_j$$

$$R_{q''}\ \vec{x} \leftarrow R_q\ \vec{y} \wedge y_i > 0 \wedge x_i = y_i - 1 \bigwedge_{j\in[1,n]\setminus\{i\}} x_j = y_j$$

Lossy channel systems and other WSTSs: Lossy counter machines are an example of a well structured transition system (WSTS) [9]. Other examples of these, such as lossy channel systems (LCSs), also have decidable reachability problems, but these do not immediately fall under our theorem because the relevant first-order theories are not decidable (in the case of LCSs the relevant theory is that of strings with concatenation with constants and the subword ordering). In some cases these results can be proved by inspecting details of exactly where in our proof the decidability property is required.

Part of our result is subsumed by the decidability of the coverability problem for WSTSs - specifically the first-order fragment where clauses only have a single foreground atom in the body and the background theory is a WQO.

C. Languages ordered by the subword order

The subword relation is a simple and important example of a WQO. [8] study the decidability of first-order theories (and fragments thereof) of languages with the subword order. Recall that a language $L \subseteq \Delta^*$ is *bounded* if $L \subseteq w_1^* \cdots w_n^*$ for some $n \geq 0$, and $w_1, \cdots, w_n \in \Delta^*$. Consider structures of the form $(L, \sqsubseteq, (w)_{w \in L})$ for some $L \subseteq \Sigma^*$ where $\sqsubseteq$ is the subword relation, and we can use every word from L as a constant.

Theorem V.1 (Kuske and Zetzsche [8])**.** *Let $L \subseteq \Delta^*$ be bounded and context free. Then the first-order theory of $(L, \sqsubseteq, (w)_{w \in L})$ is decidable.*

The theorem in fact holds for (a larger signature, and) a more expressive logic, first-order logic extended by a modulo counting quantifier [8]. The proof is by interpreting the structure in Presburger arithmetic, $(\mathbb{N}, +)$, which is known to be decidable in this logic.

Since $(L, \sqsubseteq)$ is a countable WQO, it follows from Thm. V.1 and Thm. IV.3 that the associated initial limit Datalog problem is decidable.

D. Basic process algebras and pushdown systems

An important class of countable WQO are *context-free processes* (or *basic process algebra*) and the more general collection of *pushdown systems*, with respect to the subword ordering [9]; moreover they are *automatic structures* (folklore but see e.g. [20, 21]) and so have decidable first-order theories ([22, 23] and various others). It follows that they satisfy Assumption 2.

VI. Related work and further directions

a) Decidable classes of constrained Horn clauses: Cox, McAloon and Tretkoff [4] have shown a catalogue of subrecursive complexity results for various fragments of CHC obtained by restricting the syntax (in particular, the placement of variables) and the mechanism by which parameters are passed. Our work, however, takes Kaminski, Cuenca Grau, Kostylev and Motik's limit restriction [6] as the starting point.

The limit restriction was introduced as a way of taming the undecidability of Datalog$_\mathbf{Z}$ [24] that was compatible with the desire to express problems in declarative data analysis. Moreover, it is shown in [6] that, under reasonable assumptions, the data complexity of the entailment in the logic is PTIME. Our work extends limit Datalog$_\mathbf{Z}$ to higher-orders. Higher-order extensions of Datalog are interesting in their own right: [17] have shown that, on ordered databases, order-k Datalog captures $(k-1)$-EXPTIME.

b) Decidability beyond first order: There is a lot of interest in the decidability of theories that go beyond first-order logic. A very well studied case is that of monadic second-order theories (see e.g. [25]). Of these, perhaps the best known is Rabin's celebrated result on the decidability of the theory of two successor functions [26], from which the decidability of several other monadic second-order theories can be deduced.

For applications in e.g. higher-order program verification, however, it is important to retain higher-type relations of all arities and to admit background theories. A recent work with similar requirements is that of [27] who, motivated by applications in program synthesis, have introduced the logic EQSMT. Formulas of this logic have a $\exists^*\forall^*$ prefix supporting second-order quantification at certain types. They show that satisfiability of EQSMT formulas is decidable whenever satisfiability for the relevant fragments of the background theories is decidable.

c) Higher-order constrained Horn clauses: Our work takes place in the setting of HoCHC [3]. Even when the background theory is decidable, satisfiability of HoCHC is typically undecidable (already, first-order constrained Horn is typically undecidable [24]). However [7, § VIII] identified the so-called Bernays-Schönfinkel-Ramsey fragment of HoCHC, modulo a restricted form of linear integer arithmetic, has a decidable satisfiability problem by showing equi-satisfiability to clauses w.r.t. a finite number of background theories with finite domains. (HoCHC satisfiability is decidable for trivial background theories (e.g. those of finite domains).)

An alternative higher-order logic supporting integer arithmetic is HoFL$_Z$ of [28]. Whilst we do not know of any work on decidable fragments of HoFL$_Z$, we expect that a version of our results on initial limit Datalog$_\mathbf{Z}$ could be transposed into that setting.

Future directions: One question that remains open is: for which sets of types is the higher-order limit Datalog$_\mathbf{Z}$ problem decidable when predicates are restricted to those types? There are alternatives, broadly similar to Def. IV.1, which are neither a superset nor a subset of the set of initial types, for which the same proof strategy works (and we conjecture that such results can be proved as corollaries to Thm. IV.3, by inserting dummy variables).

Except for the lower bounds due to being a superset of higher-order Datalog, we have not considered runtime complexity of this problem. If the algorithm derived from the decidability proof were used, calculating its runtime would be an exercise in the construction of large numbers. Since many practical uses would have shapes that could be converted to 1st-order programs, there is some hope for tractable performance on useful subsets of initial limit Datalog$_\mathbf{Z}$.

Conclusion: We have presented *initial limit Datalog*, the first higher-order extension of constrained Horn clauses (over a non-trivial background theory) for which the satisfiability problem is decidable. Moreover the decision procedure extends to a variety of background theories, including linear integer arithmetic, and any countable well-quasi-order with a decidable first-order theory. Our decidability proof uses a new kind of term model, called *entwined structures*, which are recursively enumerable, and model checking is decidable.

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