On the Reparameterisation Gradient for Non-Differentiable but Continuous Models

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Variational Inference:

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frame posterior inference as (deterministic) optimisation problem

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KL divergence

$$\boxed{\operatorname{argmax}_{\boldsymbol{\theta}} \mathbb{E}_{\mathbf{z} \sim q_{\boldsymbol{\theta}}} \left[f(\boldsymbol{\theta}, \mathbf{z}) \right]}$$

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Key ingredient: estimation of gradient of expectation

- **Score Estimator:**
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- **Reparametrisation Estimator:**

$$\boxed{\operatorname{argmax}_{\boldsymbol{\theta}} \mathbb{E}_{\mathbf{z} \sim q_{\boldsymbol{\theta}}} [f(\boldsymbol{\theta}, \mathbf{z})]}$$

expressed in PL with conditionals,
may *not* be *differentiable/continuous*

use ***Stochastic Gradient Descent***

Key ingredient: estimation of gradient of expectation

- **Score Estimator:**
  widely applicable but *high variance*
- **Reparametrisation Estimator:**
  better in practice but *may be biased!* [Lee et al., NeurIPS 2018]

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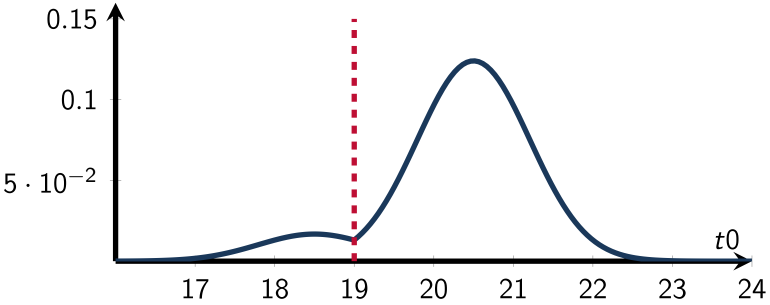

Is the Reparametrisation Gradient Estimator biased for continuous but possibly non-differentiable programs?

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No!

simply typed λ-calculus with $\mathbb{R}$ and primitive operations
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$+$ sample $+$ observe

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Unbiasedness for terms without conditionals

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Unbiasedness for terms without conditionals

Example. Rephrase conditional via non-differentiable primitive:

$$c \cdot \left(\underline{\mathrm{ReLU}}\,(19 - t_0)\right)$$

Continuity for Terms with Branching

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- efficiently check guard's consistency (linear arithmetic solvers)

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- categorical models
- prove *unbiasedness* in continuous setting
- *establish continuity* in languages with conditionals compositionally
  - for affine guards: efficient *randomised* check employing *linear arithmetic solvers*

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Foundation for *fast yet correct* (variational) inference for probabilistic programming